

SMALL VALUES OF LUSTERNIK-SCHNIRELMANN AND SYSTOLIC CATEGORIES FOR MANIFOLDS

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ABSTRACT. We prove that manifolds of Lusternik-Schnirelmann category 2 necessarily have free fundamental group. We thus settle a 1992 conjecture of Gomez-Larrañaga and Gonzalez-Acuña, by generalizing their result in dimension 3, to all higher dimensions. We examine its ramifications in systolic topology, and provide a sufficient condition for ensuring a lower bound of 3 for systolic category.

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1. INTRODUCTION

We follow the normalization of the Lusternik-Schnirelmann category (LS category) used in the recent monograph [CLOT03] (see Section 3 for a definition). Spaces of LS category 0 are contractible, while a

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closed manifold of LS category 1 is homotopy equivalent (and hence homeomorphic) to a sphere.

The characterization of closed manifolds of LS category 2 was initiated in 1992 by J. Gomez-Larrañaga and F. Gonzalez-Acuña [GG92] (see also [OR01]), who proved the following result on closed manifolds M of dimension 3. Namely, the fundamental group of M is free and nontrivial if and only if its LS category is 2. Furthermore, they conjectured that the fundamental group of every closed n -manifold, $n \geq 3$, of LS category 2 is necessarily free [GG92, Remark, p. 797]. Our interest in this natural problem was stimulated in part by recent work on the comparison of the LS category and the systolic category [KR06, KR05, Ka07], which was inspired, in turn, by M. Gromov's systolic inequalities [Gr83, Gr96, Gr99, Gr07].

In the present text we prove this 1992 conjecture. Recall that all closed surfaces different from S^2 are of LS category 2.

1.1. Theorem. *A closed connected manifold of LS category 2 either is a surface, or has free fundamental group.*

1.2. Corollary. *Every manifold M^n , $n \geq 3$, with non-free fundamental group satisfies $\text{cat}_{\text{LS}}(M) \geq 3$.*

1.3. Theorem. *Given a finitely presented group π and non-negative integer numbers k, l , there exists a closed manifold M such that $\pi_1(M) = \pi$, while $\text{cat}_{\text{LS}} M = 3 + k$ and $\dim M = 5 + 2k + l$. Furthermore, if π is not free then M can be chosen 4-dimensional with $\text{cat}_{\text{LS}} M = 3$.*

1.4. Remark. Our Theorem 1.3 indicates that Theorem 1.1, namely our generalization of [GG92], in fact optimally clarifies their result, to the extent that we identify the precise nature of the effect of the fundamental group on the LS category of manifolds.

The above results lead to the following questions:

1.5. Question. If M^4 has free fundamental group, then we have the bound $\text{cat}_{\text{LS}} M \leq 3$. Is it true that $\text{cat}_{\text{LS}} M \leq 2$?

On the side of systolic category (see below), the following result is immediate from the structure of the spaces $K(\mathbb{Z}, 1)$ and $K(\mathbb{Z}, 2)$, cf. Theorem 6.4.

1.6. Theorem. *An n -manifold with free fundamental group and torsion-free H_2 is of systolic category at most $n - 2$.*

Partly motivated by this observation, and also by the similar bound for the cuplength, we were led to the following question on the LS side.

In an earlier version of the paper, we asked whether the inequality $\text{cat}_{\text{LS}} M \leq n-2$ holds for connected n -manifolds with free fundamental group and $n > 3$. Recently J. Strom [Str07] proved that this holds for all $n > 4$, and even for CW -spaces, not only manifolds. Subsequently, we found a simple proof of this assertion for manifolds, but again with the restriction $n > 4$, see Proposition 4.3.

1.7. Question. If $\text{cat}_{\text{LS}} M = 2$, is it true that $\text{cat}_{\text{LS}}(M \setminus \{pt\}) = 1$? A direct proof would imply the main theorem trivially.

1.8. Question. Given integers m and n , describe the fundamental groups of closed manifolds M with $\dim M = n$ and $\text{cat}_{\text{LS}} M = m$.

Note that in the case $m = n$, the fundamental group of M is of cohomological dimension $\geq n$, see e.g. [CLOT03, Proposition 2.51]. Thus, we can ask when the converse holds.

1.9. Question. Given a finitely presented group π and an integer $n \geq 4$ such that $H^n(\pi) \neq 0$, when can one find a closed manifold M satisfying $\pi_1(M) = \pi$ and $\dim M = \text{cat}_{\text{LS}} M = n$? Note that Proposition 5.11 shows that such a manifold M does not always exist.

We apply Corollary 1.2 to prove that the systolic category of a 4-manifold is a lower bound for its LS category. The definition of systolic category is reviewed in Section 6.

1.10. Theorem. *Every closed orientable 4-manifold M satisfies the inequality $\text{cat}_{\text{sys}}(M) \leq \text{cat}_{\text{LS}}(M)$.*

Note that the result on cat_{LS} applies to all topological 4-manifolds. Such a manifold is homotopy equivalent to a finite CW -complex. The notion of cat_{sys} makes sense for an arbitrary finite CW -complex, as the latter is homotopy equivalent to a simplicial one, in which volumes and systoles can be defined.

We also prove the following lower bound for systolic category, which is a weak analogue of Corollary 1.2, by exploiting a technique based on Lescop's generalization of the Casson-Walker invariant [Le96, KL05]. The Abel–Jacobi map of M is reviewed in Section 7.

1.11. Theorem. *Let M be an n -manifold satisfying $b_1(M) = 2$. If the self-linking class of a typical fiber of the Abel–Jacobi map is a nontrivial class in $H_{n-3}(M)$, then $\text{cat}_{\text{sys}}(M) \geq 3$.*

The proof of the main theorem proceeds roughly as follows. If the group $\pi := \pi_1(M)$ is not free, then by a result of J. Stallings and R. Swan, the group π is of cohomological dimension at least 2. We then show that π carries a suitable nontrivial 2-dimensional cohomology

class u with twisted coefficients, and of category weight 2. Viewing M as a subspace of $K(\pi, 1)$ that contains the 2-skeleton $K(\pi, 1)^{(2)}$, and keeping in mind the fact that the 2-skeleton carries the fundamental group, we conclude that the restriction (pullback) of u to M is non-zero and also has category weight 2. By Poincaré duality with twisted coefficients, one can find a complementary $(n - 2)$ -dimensional cohomology class. By a category weight version of the cuplength argument, we therefore obtain a lower bound of 3 for $\text{cat}_{\text{LS}} M$.

In Section 2, we review the material on local coefficient systems, a twisted version of Poincaré duality, and 2-dimensional cohomology of non-free groups. In Section 3, we review the notion of category weight. In Section 4, we prove the main theorem. In Section 5 we prove Theorem 1.3. In Section 6, we recall the notion of systolic category, and prove that it provides a lower bound for the LS category of a 4-manifold. In Section 7, we recall a 1983 result of M. Gromov's and apply it to obtain a lower bound of 3 for systolic category for a class of manifolds defined by a condition of non-trivial self-linking of a typical fiber of the Abel–Jacobi map.

2. COHOMOLOGY WITH LOCAL COEFFICIENTS

A *local coefficient system* \mathcal{A} on a path connected CW -space X is a functor from the fundamental groupoid $\Gamma(X)$ of X , to the category of abelian groups. See [Ha02], [Wh78] for the definition and properties of local coefficient systems.

In other words, an abelian group \mathcal{A}_x is assigned to each point $x \in X$, and for each path α joining x to y , an isomorphism $\alpha^* : \mathcal{A}_y \rightarrow \mathcal{A}_x$ is given. Furthermore, paths that are homotopic are required to yield the same isomorphism.

Let $\pi = \pi_1(X)$, and let $\mathbb{Z}[\pi]$ be the group ring of π . Note that all the groups \mathcal{A}_x are isomorphic to a fixed group A . We will refer to A as a *stalk* of \mathcal{A} .

Given a map $f : Y \rightarrow X$ and a local coefficient system \mathcal{A} on X , we define a local coefficient system on Y , denoted $f^*\mathcal{A}$, as follows. The map f yields a functor $\Gamma(f) : \Gamma(Y) \rightarrow \Gamma(X)$, and we define $f^*\mathcal{A}$ to be the functor $\mathcal{A} \circ \Gamma(f)$. Given a pair of coefficient systems \mathcal{A} and \mathcal{B} , the tensor product $\mathcal{A} \otimes \mathcal{B}$ is defined by setting $(\mathcal{A} \otimes \mathcal{B})_x = \mathcal{A}_x \otimes \mathcal{B}_x$.

2.1. Example. A useful example of a local coefficient system is given by the following construction. Given a fiber bundle $p : E \rightarrow X$ over X , set $F_x = p^{-1}(x)$. Then the family $\{H_k(F_x)\}$ can be regarded a local coefficient system, see [Wh78, Example 3, Ch. VI, §1]. An important special case is that of an n -manifold M and spherical tangent bundle $p :$

$E \rightarrow M$ with fiber S^{n-1} , yielding a local coefficient system \mathcal{O} with $\mathcal{O}_x = H_{n-1}(S_x^{n-1}) \cong \mathbb{Z}$. This local system is called the *orientation sheaf* of M .

2.2. Remark. There is a bijection between local coefficients on X and $\mathbb{Z}[\pi]$ -modules [Sp66, Ch. 1, Exercises F]. If \mathcal{A} is a local coefficient system with stalk A , then the natural action of the fundamental group on A turns A into a $\mathbb{Z}[\pi]$ -module. Conversely, given a $\mathbb{Z}[\pi]$ -module A , one can construct a local coefficient system $\mathcal{L}(A)$ such that induced $\mathbb{Z}[\pi]$ -module structure on A coincides with the given one, cf. [Ha02].

We recall the definition of the (co)homology groups with local coefficients via modules [Ha02]:

$$(2.1) \quad H^k(X; \mathcal{A}) \cong H^k(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{X}), A), \delta)$$

and

$$(2.2) \quad H_k(X; \mathcal{A}) \cong H_k(A \otimes_{\mathbb{Z}[\pi]} C_*(\tilde{X}), 1 \otimes \partial).$$

Here $(C_*(\tilde{X}), \partial)$ is the chain complex of the universal cover \tilde{X} of X , A is the stalk of the local coefficient system \mathcal{A} , and δ is the coboundary operator. Note that in the tensor product we used the right $\mathbb{Z}[\pi]$ module structure on A defined via the standard rule $ag = g^{-1}a$, for $a \in A, g \in \pi$.

Recall that for CW -complexes X , there is a natural bijection between equivalence classes of local coefficient systems and locally constant sheaves on X . One can therefore define (co)homology with local coefficients as the corresponding sheaf cohomology [Bre97]. In particular, we refer to [Bre97] for the definition of the cup product

$$\cup : H^i(X; \mathcal{A}) \otimes H^j(X; \mathcal{B}) \rightarrow H^{i+j}(X; \mathcal{A} \otimes \mathcal{B})$$

and the cap product

$$\cap : H_i(X; \mathcal{A}) \otimes H^j(X; \mathcal{B}) \rightarrow H_{i-j}(X; \mathcal{A} \otimes \mathcal{B}).$$

A nice exposition of the cup and the cap products in a slightly different setting can be found in [Bro94]. In particular, we have the cap product

$$H_k(X; \mathcal{A}) \otimes H^k(X; \mathcal{B}) \rightarrow H_0(X; \mathcal{A} \otimes \mathcal{B}) \cong A \otimes_{\mathbb{Z}[\pi]} B.$$

2.3. Proposition. *Given an integer $k \geq 0$, there exists a local coefficient system \mathcal{B} and a class $v \in H^k(X; \mathcal{B})$ such that, for every local coefficient system \mathcal{A} and nonzero class $a \in H_k(X; \mathcal{A})$, we have $a \cap v \neq 0$.*

Proof. We convert the stalk of \mathcal{A} into a right $\mathbb{Z}[\pi]$ -module A as above. We use the isomorphisms (2.1) and (2.2). Consider the chain $\mathbb{Z}[\pi]$ -complex

$$\dots \longrightarrow C_{k+1}(\tilde{X}) \xrightarrow{\partial_{k+1}} C_k(\tilde{X}) \xrightarrow{\partial_k} C_{k-1}(\tilde{X}) \longrightarrow \dots$$

For the given k , we set $B := C_k(\tilde{X}) / \text{Im } \partial_{k+1}$. Let \mathcal{B} be the corresponding local system on X . Thus, we obtain the exact sequence of $\mathbb{Z}[\pi]$ -modules

$$C_{k+1}(\tilde{X}) \xrightarrow{\partial_{k+1}} C_k(\tilde{X}) \xrightarrow{f} B \rightarrow 0.$$

Note that the epimorphism f can be regarded as a k -cocycle with values in \mathcal{B} , since $\delta f(x) = f\partial_{k+1}(x) = 0$. Let $v := [f] \in H^k(X; \mathcal{B})$ be the cohomology class of f . Now we prove that

$$a \cap [f] \neq 0.$$

Since the tensor product is right exact, we obtain the diagram

$$\begin{array}{ccccccc} A \otimes_{\mathbb{Z}[\pi]} C_{k+1}(\tilde{X}) & \xrightarrow{1 \otimes \partial_{k+1}} & A \otimes_{\mathbb{Z}[\pi]} C_k(\tilde{X}) & \xrightarrow{1 \otimes f} & A \otimes_{\mathbb{Z}[\pi]} B & \longrightarrow & 0 \\ & & & & \downarrow g & & \\ & & & & A \otimes_{\mathbb{Z}[\pi]} C_{k-1}(\tilde{X}) & & \end{array}$$

where the row is exact. The composition

$$A \otimes_{\mathbb{Z}[\pi]} C_k(\tilde{X}) \xrightarrow{1 \otimes f} A \otimes_{\mathbb{Z}[\pi]} B \xrightarrow{g} A \otimes_{\mathbb{Z}[\pi]} C_{k-1}(\tilde{X})$$

coincides with $1 \otimes \partial_k$. We represent the class a by a cycle

$$z \in A \otimes_{\mathbb{Z}[\pi]} C_k(\tilde{X}).$$

Since $z \notin \text{Im}(1 \otimes \partial_{k+1})$, we conclude that

$$(1 \otimes f)(z) \neq 0 \in A \otimes_{\mathbb{Z}[\pi]} B = H_0(X; \mathcal{A} \otimes \mathcal{B}).$$

Thus, for the cohomology class v of f we have $a \cap v \neq 0$. \square

Every closed connected n -manifold M satisfies $H_n(M; \mathcal{O}) \cong \mathbb{Z}$. A generator (one of two) of this group is called the *fundamental class* of M and is denoted by $[M]$.

One has the following generalization of the Poincaré duality isomorphism.

2.4. Theorem ([Bre97]). *The homomorphism*

$$(2.3) \quad \Delta : H^i(M; \mathcal{A}) \rightarrow H_{n-i}(M; \mathcal{O} \otimes \mathcal{A})$$

defined by setting $\Delta(a) = a \cap [M]$, is an isomorphism.

In fact, in [Bre97] there is the sheaf \mathcal{O}^{-1} at the right, but for manifolds we have $\mathcal{O} = \mathcal{O}^{-1}$.

Given a group π and a $\mathbb{Z}[\pi]$ -module A , we denote by $H^*(\pi; A)$ the cohomology of the group π with coefficients in A , see e.g. [Bro94]. Recall that $H^i(\pi; A) = H^i(K(\pi, 1); \mathcal{L}(A))$, see Remark 2.2.

Let F be a principal ideal domain and let $\text{cd}_F(\pi)$ denote the cohomological dimension of π over F , i.e. the largest m such that there exists an $F[\pi]$ -module A with $H^m(\pi; A) \neq 0$.

2.5. Theorem ([Sta68, Swan69]). *If $\text{cd}_{\mathbb{Z}} \pi \leq 1$ then π is a free group.*

We need the following well-known fact.

2.6. Lemma. *If π be a group with $\text{cd}_{\mathbb{Z}} \pi = q \geq 2$. Then $H^2(\pi; A) \neq 0$ for some $\mathbb{Z}[\pi]$ -module A .*

Proof. Let $0 \rightarrow A' \rightarrow J \rightarrow A'' \rightarrow 0$ be an exact sequence of $\mathbb{Z}[\pi]$ -modules with J injective. Then $H^k(\pi; A') = H^{k-1}(\pi; A'')$ for $k > 1$. Since $H^q(\pi; B) \neq 0$ for some B , the proof can be completed by an obvious induction. \square

2.7. Remark. Let $u \in H^1(\pi; I(\pi))$ be the Berstein-Švarc class described in [CLOT03, Proposition 2.51]. If $3 \leq \text{cd}_{\mathbb{Z}}(\pi) = n < \infty$ then $\text{cat}_{\text{LS}}(K(\pi, 1)) = n = \dim K(\pi, 1)$ by [EG57], and hence $u^{\otimes n} \neq 0$ by [CLOT03, Proposition 2.51] (for $n = \infty$ this means that $u^{\otimes k} \neq 0$ for all k). In particular, $H^2(\pi; I(\pi) \otimes I(\pi)) \neq 0$. However, we do not know if $u \otimes u \neq 0$ in case $\text{cd}_{\mathbb{Z}}(\pi) = 2$.

3. CATEGORY WEIGHT AND LOWER BOUNDS FOR cat_{LS}

3.1. Definition ([BG61, Fe53, Fo41]). Let $f : X \rightarrow Y$ be a map of (locally contractible) CW -spaces. The *Lusternik–Schnirelmann category of f* , denoted $\text{cat}_{\text{LS}}(f)$, is defined to be the minimal integer k such that there exists an open covering $\{U_0, \dots, U_k\}$ of X with the property that each of the restrictions $f|_{A_i} : A_i \rightarrow Y$, $i = 0, 1, \dots, k$ is null-homotopic.

The *Lusternik–Schnirelmann category $\text{cat}_{\text{LS}} X$ of a space X* is defined as the category $\text{cat}_{\text{LS}}(1_X)$ of the identity map.

3.2. Definition. The *category weight* $\text{wgt}(u)$ of a non-zero cohomology class $u \in H^*(X; \mathcal{A})$ is defined as follows:

$$\text{wgt}(u) \geq k \iff \{\varphi^*(u) = 0 \text{ for every } \varphi : F \rightarrow X \text{ with } \text{cat}_{\text{LS}}(\varphi) < k\}.$$

3.3. Remark. E. Fadell and S. Husseini (see [FH92]) originally proposed the notion of category weight. In fact, they considered an invariant similar to the wgt of (3.2) (denoted in [FH92] by cwgt), but

where the defining maps $\varphi: F \rightarrow X$ were required to be inclusions rather than general maps. As a consequence, cwgt is not a homotopy invariant, and thus a delicate quantity in homotopy calculations. Yu. Rudyak [Ru97, Ru99] and J. Strom [Str97] suggested the homotopy invariant version of category weight as defined in Definition 3.2. Rudyak called it *strict* category weight (using the notation $\text{swgt}(u)$) and Strom called it *essential* category weight (using the notation $E(u)$). At the Mt. Holyoke conference in 2001, both creators agreed to adopt the notation wgt and call it simply *category weight*.

3.4. Proposition ([Ru97, Str97]). *Category weight has the following properties.*

- (1) $1 \leq \text{wgt}(u) \leq \text{cat}_{\text{LS}}(X)$, for all $u \in \tilde{H}^*(X; \mathcal{A})$, $u \neq 0$.
- (2) For every $f: Y \rightarrow X$ and $u \in H^*(X; \mathcal{A})$ with $f^*(u) \neq 0$ we have $\text{cat}_{\text{LS}}(f) \geq \text{wgt}(u)$ and $\text{wgt}(f^*(u)) \geq \text{wgt}(u)$.
- (3) For $u \in H^*(X; \mathcal{A})$ and $v \in H^*(X; \mathcal{B})$ we have

$$\text{wgt}(u \cup v) \geq \text{wgt}(u) + \text{wgt}(v).$$

- (4) For every $u \in H^s(K(\pi, 1); \mathcal{A})$, $u \neq 0$, we have $\text{wgt}(u) \geq s$.

Proof. See [CLOT03, §2.7 and Proposition 8.22], the proofs in loc. cit. can be easily adapted to local coefficient systems. \square

4. MANIFOLDS OF LS CATEGORY 2

In this section we prove that the fundamental group of a closed connected manifold of LS category 2 is free.

4.1. Theorem. *Let M be a closed connected manifold of dimension $n \geq 3$. If the group $\pi := \pi_1(M)$ is not free, then $\text{cat}_{\text{LS}} M \geq 3$.*

Proof. By Theorem 2.5 and Lemma 2.6, there a local coefficient system \mathcal{A} on $K(\pi, 1)$ such that $H^2(K(\pi, 1); \mathcal{A}) \neq 0$. Choose a non-zero element $u \in H^2(K(\pi, 1); \mathcal{A})$. Let $f: M \rightarrow K(\pi, 1)$ be the map that induces an isomorphism of fundamental groups, and let $i: K \rightarrow M$ be the inclusion of the 2-skeleton. (If M is not triangulable, we take i to be any map of a 2-polyhedron that induces an isomorphism of fundamental groups.) Then

$$(fi)^*: H^2(K(\pi, 1); \mathcal{A}) \rightarrow H^2(K; (fi)^*\mathcal{A})$$

is a monomorphism. In particular, we have $f^*u \neq 0$ in $H^2(M; (f)^*\mathcal{A})$. Now consider the class

$$a = [M] \cap f^*u \in H_{n-2}(M; \mathcal{O}^{-1} \otimes f^*\mathcal{A}).$$

Then $a \neq 0$ by Theorem 2.4. Hence, by Proposition 2.3, there exists a class $v \in H^{n-2}(M; \mathcal{B})$ such that $a \cap v \neq 0$. We claim that $f^*u \cup v \neq 0$. Indeed, one has

$$[M] \cap (f^*u \cup v) = ([M] \cap f^*u) \cap v = a \cap v \neq 0.$$

Now, $\text{wgt } f^*u \geq 2$ by Proposition 3.4, items (2) and (4). Furthermore, $\text{wgt}(v) \geq 1$ by Proposition 3.4, item (1). We therefore obtain the lower bound $\text{wgt}(f^*u \cup v) \geq 3$ by Proposition 3.4, item (3). Since $f^*u \cup v \neq 0$, we conclude that $\text{cat}_{\text{LS}} M \geq 3$ by Proposition 3.4, item (1). \square

4.2. Corollary. *If $M^n, n \geq 3$ is a closed manifold with $\text{cat}_{\text{LS}} M \leq 2$, then $\pi_1(M)$ is a free group.*

The following Proposition is a special case of [Str07, Corollary 2]. Here we give a relatively simple geometric proof.

4.3. Proposition. *Let M be a closed connected n -dimensional PL manifold, $n > 4$, with free fundamental group. Then $\text{cat}_{\text{LS}} M \leq n - 2$.*

Proof. If X is a 2-dimensional (connected) CW-complex with free fundamental group then $\text{cat}_{\text{LS}} X \leq 1$, see e.g. [KRS06, Theorem 12.1]. Hence, if Y is a k -dimensional complex with free fundamental group then $\text{cat}_{\text{LS}} Y \leq k - 1$ for $k > 2$. Now, let K be a triangulation of M , and let L be its dual triangulation. Then $M \setminus L^{(l)}$ is homotopy equivalent to $K^{(k)}$ whenever $k + l + 1 = n$. Hence,

$$\text{cat}_{\text{LS}} M \leq \text{cat}_{\text{LS}} K^{(k)} + \text{cat}_{\text{LS}} L^{(l)} + 1.$$

Since $\pi_1(K)$ and $\pi_1(L)$ are free, we conclude that $\text{cat}_{\text{LS}} K^{(k)} \leq k - 1$ and $\text{cat}_{\text{LS}} L^{(l)} \leq l - 1$ for $k, l > 1$. Thus $\text{cat}_{\text{LS}} M \leq k - 1 + l - 1 + 1 = n - 2$. \square

5. MANIFOLDS OF HIGHER LS CATEGORY

5.1. Definition. A CW-space X is called k -essential, $k > 1$ if for every CW-complex structure on X there is no map $f : X^{(k)} \rightarrow K(\pi, 1)^{(k-1)}$ that induces an isomorphism of the fundamental groups.

5.2. Theorem. *For every closed k -essential manifold M with $\dim M > k$ we have $\text{cat}_{\text{LS}} M \geq k + 1$.*

Proof. Let M be k -essential.

Let $k = 2$. If $\text{cat}_{\text{LS}} M \leq k$, then, by Theorem 4.1, $\pi_1(M)$ is free. Hence there is a map $f : M \rightarrow \vee S^1$ that induces an isomorphism of the fundamental groups. Thus, M cannot be 2-essential.

Let $k \geq 3$. Let $K = K(\pi_1(M), 1)$. Take a map $f : M^{(k-1)} \rightarrow K^{(k-1)}$ such that the restriction $f|_{M^{(2)}}$ is the identity homeomorphism of the

2-skeleta $M^{(2)}$ and $K^{(2)}$. We consider the problem of extension of f to M .

We claim that the first obstruction $o(f) \in H^k(M; E)$ (taken with coefficients in a local system E with the stalk $\pi_{k-1}(K^{(k-1)})$) to the extension is not equal to zero.

Indeed, if $o(f) = 0$, then there exists a map $\bar{f} : M^{(k)} \rightarrow K^{(k-1)}$ which coincides with f on the $(k-2)$ -skeleton. The map $\bar{f}_* : \pi_1(M^{(k)}) \rightarrow \pi_1(K^{(k-1)})$ can be regarded as an endomorphism of $\pi_1(M)$ that is identical on generators, and therefore \bar{f}_* is an isomorphism. Hence, M is not k -essential.

Consider the commutative diagram

$$\begin{array}{ccccc} M^{(k-1)} & \xrightarrow{f} & K^{(k-1)} & \xrightarrow{\text{id}} & K^{(k-1)} \\ i \downarrow & & j \downarrow & & \\ M & \xrightarrow{\tilde{f}} & K & & \end{array}$$

where i and j are the inclusions of the skeleta. Let α be the first obstruction to the extension of id to a map $K \rightarrow K^{(k-1)}$. By commutativity of the above diagram, we have $o(f) = \tilde{f}^*(\alpha)$. Now, asserting as in the proof of Theorem 4.1, we get that $\tilde{f}^*(\alpha) \cup v \neq 0$ for some v with $\dim v = \dim M - k$. Since $\dim M > k$, we conclude that $\dim v \geq 1$ and thus $\text{cat}_{\text{LS}} M \geq k + 1$. \square

5.3. Remark. If a closed manifold M^n is n -essential then $\text{cat}_{\text{LS}} M = n$, see e.g. [KR06] and [Ka07, Theorem 12.5.2].

5.4. Proposition. *For every non-free finitely presented group π , there exists a closed 4-dimensional manifold M with fundamental group π and $\text{cat}_{\text{LS}} M = 3$.*

Proof. Take an embedding of a 2-skeleton of $K(\pi, 1)$ in \mathbb{R}^5 and let M be the boundary of the regular neighborhood of this skeleton. Then, clearly, $\pi_1(M) = \pi$. Furthermore, M admits a retraction onto its 2-skeleton. Therefore M is not 4-essential, and hence $\text{cat}_{\text{LS}} M = 3$. \square

Let M_f be the mapping cylinder of $f : X \rightarrow Y$. We use the notation $\pi_*(f) = \pi_*(M_f, X)$. Then $\pi_i(f) = 0$ for $i \leq n$ amounts to saying that it induces isomorphisms $f_* : \pi_i(X_1) \rightarrow \pi_i(Y_1)$ for $i \leq n$ and an epimorphism in dimension $n+1$. Similar notation $H_*(f) = H_*(M_f, X)$ we use for homology.

5.5. Lemma. *Let $f_j : X_j \rightarrow Y_j$ be a family of maps of CW spaces such that $H_i(f_j) = 0$ for $i < n_j$. Then $H_i(f_1 \wedge \cdots \wedge f_s) = 0$ for $i \leq \min\{n_j\}$.*

Proof. Note that $M(f_1 \wedge \cdots \wedge f_s) \cong Y_1 \wedge \cdots \wedge Y_s \cong M(f_1) \wedge \cdots \wedge M(f_s)$. Now, by using the Künneth formula and considering the homology exact sequence of the pair $(M(f_1) \wedge \cdots \wedge M(f_s), X_1 \wedge \cdots \wedge X_s)$ we get the result. \square

5.6. Proposition. *Let $f_j : X_j \rightarrow Y_j$, $3 \leq j \leq s$ be a family of maps of CW spaces such that $\pi_i(f_j) = 0$ for $i < n_j$. Then the joins satisfy*

$$\pi_k(f_1 * f_2 * \cdots * f_s) = 0$$

for $k \leq \min\{n_j\} + s - 1$.

Proof. By the version of the Relative Hurewicz Theorem for non-simply connected X_j [Ha02, Theorem 4.37], we obtain $H_i(f_j) = 0$ for $i < n_j$. By Lemma 5.5 we obtain that $H_k(f_1 \wedge \cdots \wedge f_s) = 0$ for $k \leq \min\{n_j\}$. Since the join $A_1 * \cdots * A_s$ is homotopy equivalent to the iterated suspension $\Sigma^{s-1}(A_1 \wedge \cdots \wedge A_s)$ over the smash product, we conclude that $H_k(f_1 * \cdots * f_s) = 0$ for $k \leq \min\{n_j\} + s - 1$. Since $X_1 * \cdots * X_s$ is simply connected for $s \geq 3$, by the standard Relative Hurewicz Theorem we obtain that $\pi_k(f_1 * \cdots * f_s) = 0$ for $k \leq \min\{n_j\} + s - 1$. \square

Given two maps $f : Y_1 \rightarrow X$ and $g : Y_2 \rightarrow X$, we set

$$Z = \{(y_1, y_2, t) \in Y_1 * Y_2 \mid f(y_1) = g(y_2)\}$$

and define the *fiberwise join*, or *join over X* of f and g as the map

$$f *_X g : Z \rightarrow X, \quad (f *_X g)(y_1, y_2, t) = f(y_1)$$

Let $p_0^X : PX \rightarrow X$ be the Serre path fibration. This means that PX is the space of paths on X that start at the base point of the pointed space X , and $p_0(\alpha) = \alpha(1)$. We denote by $p_n^X : G_n(X) \rightarrow X$ the n -fold fiberwise join of p_0 .

The proof of the following theorem can be found in [CLOT03].

5.7. Theorem (Ganea, Švarc). *For a CW-space X , $\text{cat}_{\text{LS}}(X) \leq n$ if and only if there exists a section of $p_n : G_n(X) \rightarrow X$.*

5.8. Proposition. *The connected sum $S^k \times S^l \# \cdots \# S^k \times S^l$ is a space of LS-category 2.*

Proof. This can be deduced from a general result of K. Hardy [H73] because the connected sum of two manifolds can be regarded as the double mapping cylinder. Alternatively, one can note that, after removing a point, the manifold on hand is homotopy equivalent to the wedge of spheres. \square

5.9. Theorem. *For every finitely presented group π and $n \geq 5$, there is a closed n -manifold M of LS-category 3 with $\pi_1(M) = \pi$.*

Proof. If the group π is the free group of rank s , we let M' be the k -fold connected sum $S^1 \times S^2 \# \cdots \# S^1 \times S^2$. Then M' is a closed 3-manifold of LS category 2 with $\pi_1(M') = F_s$. Then the product manifold $M = M' \times S^{n-3}$ has cuplength 3 and is therefore the desired manifold.

Now assume that the group π is not free. We fix a presentation of π with s generators and r relators. Let M' be the k -fold connected sum $S^1 \times S^{n-1} \# \cdots \# S^1 \times S^{n-1}$. Then M' is a closed n -manifold of the category 2 with $\pi_1(M') = F_s$. For every relator w we fix a nicely imbedded circle $S_w^1 \subset M'$ such that $S_w^{-1} \cap S_v^{-1} = \emptyset$ for $w \neq v$. Then we perform the surgery on these circles to obtain a manifold M . Clearly, $\pi_1(M) = \pi$. We show that $\text{cat}_{\text{LS}}(M) \leq 3$, and so $\text{cat}_{\text{LS}} M = 3$ by Theorem 4.1.

As usual, the surgery process yields an $(n+1)$ -manifold X with $\partial X = M \sqcup M'$. Here X is the space obtained from $M' \times I$ by attaching handles $D^2 \times D^{n-1}$ of index 2 to $M' \times 1$ along the above circles. We note that $\text{cat}_{\text{LS}}(X) \leq 3$.

On the other hand, by duality, X can be obtained from $M \times I$ by attaching handles of index $n-1$ to the boundary component of $M \times I$. In particular, the inclusion $f : M \rightarrow X$ induces an isomorphism of the homotopy groups of dimension $\leq n-3$ and an epimorphism in dimension $n-2$. Hence $\Omega f : \Omega M \rightarrow \Omega X$ induces isomorphisms in dimensions $\leq n-4$ and an epimorphism in dimension $n-3$. Thus, $\pi_i(\Omega f) = 0$ for $i \leq n-4$.

In order to prove that $\text{cat}_{\text{LS}} M \leq 3$ it suffices to show that the Ganea-Švarc fibration $p_3 : G_3(M) \rightarrow M$ has a section. Consider the commutative diagram

$$\begin{array}{ccccc} G_3 M & \xrightarrow{q} & Z & \xrightarrow{f'} & G_3(X) \\ p_M^3 \downarrow & & p' \downarrow & & \downarrow p_3^X \\ M & \xlongequal{\quad} & M & \xrightarrow{f} & X \end{array}$$

where the right-hand square is the pull-back diagram and $f'q = G_3(f)$. Note that q is uniquely determined. Since $\text{cat}_{\text{LS}}(X) \leq 3$, by Theorem 5.7 there is a section $s : X \rightarrow G_3(X)$. It defines a section $s' : M \rightarrow Z$ of p' . It suffices to show that the map $s' : M \rightarrow Z$ admits a homotopy lifting $h : M \rightarrow G_3 M$ with respect to q , i.e. the map h with $qh \cong s'$. Indeed, we have

$$p_M^3 h = p' q h \cong p' s' = 1_M$$

and so h is a homotopy section of p_3^M . Since the latter is a Serre fibration, the homotopy lifting property yields an actual section.

Let F_1 and F_2 be the fibers of fibrations p_3^M and p' , respectively. Consider the commutative diagram generated by the homotopy exact sequences of the Serre fibrations p_3^M and p' :

$$\begin{array}{ccccccc}
 \pi_i(F_1) & \longrightarrow & \pi_i(G_3(M)) & \xrightarrow{(p_3^M)_*} & \pi_i(M) & \longrightarrow & \pi_{i-1}(F_1) \longrightarrow \cdots \\
 \downarrow \phi_* & & \downarrow q_* & & \downarrow = & & \downarrow \phi_* \\
 \pi_i(F_2) & \longrightarrow & \pi_i(Z) & \xrightarrow{(p')_*} & \pi_i(M) & \longrightarrow & \pi_{i-1}(F_2) \longrightarrow \cdots
 \end{array}$$

Note that we have

$$\phi = \Omega(f) * \Omega(f) * \Omega(f) * \Omega(f).$$

By Proposition 5.6 and since $\pi_i(\Omega f) = 0$ for $i \leq n - 5$, we conclude that $\pi_i(\phi) = 0$ for $i \leq n - 4 + 3 = n - 1$. Hence ϕ induces an isomorphism of the homotopy groups of dimensions $\leq n - 1$ and an epimorphism in dimension n . By the Five Lemma we obtain that q_* is an isomorphism in dimensions $\leq n - 1$ and an epimorphism in dimension n . Hence the homotopy fiber of q is $(n - 1)$ -connected. Since $\dim M = n$, the map s' admits a homotopy lifting $h : M \rightarrow G_3(M)$. \square

5.10. Corollary. *Given a finitely presented group π and non-negative integer numbers k, l there exists a closed manifold M such that $\pi_1(M) = \pi$, while $\text{cat}_{\text{LS}} M = 3 + k$ and $\dim M = 5 + 2k + l$.*

Proof. By Theorem 5.9, there exists a manifold N such that $\pi_1(M) = \pi$, $\text{cat}_{\text{LS}} M = 3$ and $\dim M = 5 + l$. Moreover, this manifold N possesses a detecting element, i.e. a cohomology class whose category weight is equal to $\text{cat}_{\text{LS}} N = 3$. For π free this follows since the cuplength of N is equal to 3, for other groups we have the detecting element $f^*u \cup v$ constructed in the proof of Theorem 4.1. If a space X possesses a detecting element then, for every $k > 0$, we have $\text{cat}_{\text{LS}}(X \times S^k) = \text{cat}_{\text{LS}} X + 1$ and $X \times S$ possesses a detecting element, [Ru99]. Now, the manifold $M := N \times (S^2)^k$ is the desired manifold. \square

Generally, we have a question about relations between the category, the dimension, and the fundamental group of a closed manifold. The following proposition shows that the situation quite intricate.

5.11. Proposition. *Let p be an odd prime. Then there exists a closed $(2n + 1)$ -manifold with $\text{cat}_{\text{LS}} M = \dim M$ and $\pi_1(M) = \mathbb{Z}_p$, but there are no closed $2n$ -manifolds with $\text{cat}_{\text{LS}} M = \dim M$ and $\pi_1(M) = \mathbb{Z}_p$.*

Proof. An example of $(2n + 1)$ -manifold is the quotient space S^{2n+1}/\mathbb{Z}_p with respect to a free \mathbb{Z}_p -action on S^{2n+1} . Now, given a $2n$ -manifold with $\pi_1(M) = \mathbb{Z}_p$, consider a map $f : M \rightarrow K(\mathbb{Z}_p, 1)$ that induces

an isomorphism of fundamental groups. Since $H_{2n}(K(\mathbb{Z}_p, 1)) = 0$, it follows from the obstruction theory and Poincaré duality that f can be deformed into the $(2n - 1)$ -skeleton of $K(\mathbb{Z}_p, 1)$, cf. [Ba93, Section 8]. Hence, M is inessential, and thus $\text{cat}_{\text{LS}} M < 2n$ [KR06]. \square

6. UPPER BOUND FOR SYSTOLIC CATEGORY

In this section, we recall the definition of systolic category cat_{sys} , and prove an upper bound for cat_{sys} of a smooth manifold with free fundamental group. Combined with the result of the previous section, we thus obtain a proof of Theorem 1.10.

The main idea behind the definition of systolic category is to bound the total (top-dimensional) volume from below by a product of lower-dimensional systolic information, in the following precise sense, cf. (6.2).

6.1. Definition. Let X be a Riemannian manifold with the Riemannian metric \mathcal{G} . Given $k \in \mathbb{N}$, $k > 1$, we set

$$\text{sys}_k(X, \mathcal{G}) = \inf\{\text{sysh}_k(X, \mathcal{G}; \mathbb{Z}[B]), \text{sysh}_k(X, \mathcal{G}; \mathbb{Z}_2[B]), \text{stsys}_k(X, \mathcal{G})\},$$

where the infimum is over all groups B of regular covering spaces of X . Note that a systole of X with coefficients in the group ring of B is by definition the systole of the corresponding covering space $\bar{X} \rightarrow X$ with deck group B . Here sysh_k is the homology k -systole, while $\mathbb{Z}[B]$ is the \mathbb{Z} -group ring of B and similarly for \mathbb{Z}_2 . The invariant stsys_k is the stable k -systole. See [KR06] and [Ka07, Chapter 12] for more detailed definitions. Furthermore, we define

$$\text{sys}_1(X, \mathcal{G}) = \min\{\text{sys}\pi_1(X, \mathcal{G}), \text{stsys}_1(X, \mathcal{G})\}.$$

Note that the systolic invariants thus defined are positive (or infinite), see [KR05].

Let X be an n -dimensional polyhedron, and let $d \geq 1$ be an integer. Consider a partition

$$(6.1) \quad n = k_1 + \dots + k_d,$$

where $k_i \geq 1$ for all $i = 1, \dots, d$. We will consider scale-invariant inequalities “of length d ” of the following type:

$$(6.2) \quad \text{sys}_{k_1}(\mathcal{G}) \text{sys}_{k_2}(\mathcal{G}) \dots \text{sys}_{k_d}(\mathcal{G}) \leq C(X) \text{vol}_n(\mathcal{G}),$$

satisfied by all metrics \mathcal{G} on X , where the constant $C(X)$ is expected to depend only on the topological type of X , but not on the metric \mathcal{G} . Here the quantity sys_k denotes the infimum of all systolic invariants in dimension k , as defined above.

6.2. Definition. Systolic category of X , denoted $\text{cat}_{\text{sys}}(X)$, is the largest integer d such that there exists a partition (6.1) with

$$\prod_{i=1}^d \text{sys}_{k_i}(X, \mathcal{G}) \leq C(X) \text{vol}_n(X, \mathcal{G})$$

for all metrics \mathcal{G} on X . If no such partition and inequality exist, we define systolic category to be zero.

In particular, we have $\text{cat}_{\text{sys}} X \leq \dim X$.

6.3. Remark. Clearly, systolic category equals 1 if and only if the polyhedron possesses an n -dimensional homology class, but the volume cannot be bounded from below by products of systoles of positive codimension. Systolic category vanishes if X is contractible.

6.4. Theorem. *Let M be an n -manifold, $n \geq 4$, with $H_2(M)$ torsion-free. Suppose the fundamental group of M is free. Then its systolic category is at most $n - 2$.*

Proof. By hypothesis, we have $H_2(M) = \mathbb{Z}^{b_2}$. Consider a map

$$M \rightarrow K(\pi, 1) \times K(\mathbb{Z}^{b_2}, 2),$$

inducing an isomorphism of fundamental groups, as well as isomorphism of 2-dimensional homology. Here the first factor is a wedge of circles, while the second is a product of $b_2(M)$ copies of \mathbb{CP}^∞ . We work with the product cell structure. We may assume that the image of M lies in the n -skeleton. We consider separately two cases according to the parity of the dimension.

If n is even, then all cells involving circles from the first factor have positive codimension. Since the map in homology is injective, we can apply pullback arguments for metrics as in [BKSW06]. Therefore we can obtain metrics with fixed volume and fixed sys_2 , but with arbitrarily large sys_1 . In more detail, the pullback arguments for metrics originate in I. Babenko's 1992 paper translated in [Ba93], where he treats the case of the 1-systoles. In [Ba06], he describes the argument for the stable 1-systole. In [BKSW06], we present an argument that treats the case of the stable k -systole, involving a suitable application of the coarea formula and an isoperimetric inequality.

This precludes the possibility of a lower bound for the total volume corresponding to a partition of type (6.1) satisfying

$$\max_i k_i = 2,$$

i.e. involving only the 1-systole and the 2-systole. Thus any lower bound must involve a k -systole with $k \geq 3$, and therefore we obtain $\text{cat}_{\text{sys}} \leq n - 2$, proving the theorem for even n .

If $n \geq 5$ is odd, each top dimensional cell is a copy of $S^1 \times \mathbb{CP}^{(n-1)/2}$. By considering the product metric of a circle of length $L \rightarrow \infty$ with a fixed metric on $\mathbb{CP}^{(n-1)/2}$, we see that both the volume and the 1-systole grow linearly in L . Therefore a product of the form

$$(\text{sys}_1)^a (\text{sys}_2)^b$$

can only be a lower bound for the volume if the first exponent satisfies $a = 1$. Thus a factor of either $(\text{sys}_2)^2$ or sys_k with $k \geq 3$ must be involved in (6.2), and again we obtain $\text{cat}_{\text{sys}} \leq n - 2$. \square

6.5. Corollary. *The systolic category of an orientable 4-manifold with free fundamental group is at most 2.*

Proof. If the fundamental group is free, then by the universal coefficient theorem, 2-dimensional cohomology is torsion-free. By \mathbb{Z} -Poincaré duality, the same is true of homology, and we apply the previous theorem. \square

Proof of Theorem 1.10. Let M be an orientable 4-manifold with free fundamental group. The possible inequalities of type (6.2) in dimension 4 correspond to one of the three partitions $4 = 1 + 3$, $4 = 2 + 2$, and $4 = 1 + 1 + 2$. The latter is ruled out by the previous theorem. In either of the first two cases, we have $\text{cat}_{\text{sys}} = 2$. Since a 4-manifold with $\text{cat}_{\text{sys}} = 4$ must be essential with suitable coefficients [Ba93, Gr83, KR06], it follows that $\text{cat}_{\text{sys}} \leq \text{cat}_{\text{LS}}$ for all orientable 4-manifolds. \square

7. SELF-LINKING OF FIBERS AND A LOWER BOUND FOR cat_{sys}

There are three main constructions for obtaining systolic lower bounds for the total volume of a closed manifold M . All three originate in Gromov's 1983 Filling paper [Gr83], and can be summarized as follows.

- (1) Gromov's inequality for the homotopy 1-systole of an essential manifold M , see [We05, Gu06, Bru07] and [Ka07, p. 97].
- (2) Gromov's stable systolic inequality (treated in more detail in [BK03, BK04]) corresponding to a cup product decomposition of the rational fundamental cohomology class of M .
- (3) A construction using the Abel–Jacobi map to the Jacobi torus of M (sometimes called the dual torus), also based on a theorem of Gromov (elaborated in [IK04, BCIK07]).

Let us describe the last construction in more detail. Let M be a connected n -manifold. Let $b = b_1(M)$. Let

$$\mathbb{T}^b := H_1(M; \mathbb{R}) / H_1(M; \mathbb{Z})_{\mathbb{R}}$$

be its Jacobi torus (sometimes called the dual torus). A natural metric on the Jacobi torus of a Riemannian manifold is defined by the stable norm, see [Ka07, p. 94].

The Abel–Jacobi map $\mathcal{A}_M : M \rightarrow \mathbb{T}^b$ is discussed in [Li69, BK04], cf. [Ka07, p. 139]. A typical fiber $F_M \subset M$ (i.e. inverse image of a generic point) of \mathcal{A}_M is a smooth imbedded $(n-b)$ -submanifold (varying in type as a function of the point of \mathbb{T}^b). Our starting point is the following observation of Gromov’s [Gr83, Theorem 7.5.B], elaborated in [IK04].

7.1. Theorem (M. Gromov). *If the homology class $[\overline{F}_M] \in H_{n-b}(M)$ of the lift of F_M to the maximal free abelian cover of M is nonzero, then the total volume of M admits a lower bound in terms of the product of the volume of the Jacobi torus and the infimum of areas of cycles representing $[\overline{F}_M]$.*

7.2. Proposition. *If a typical fiber of the Abel–Jacobi map represents a nontrivial $(n-b)$ -dimensional homology class in M , then systolic category satisfies $\text{cat}_{\text{sys}}(M) \geq b+1$.*

Proof. If the fiber class is nonzero, then the Abel–Jacobi map is necessarily surjective in the set-theoretic sense. One then applies the technique of Gromov’s proof of Theorem 7.1, cf. [IK04], combined with a lower bound for the volume of the Jacobi torus in terms of the b -th power of the stable 1-systole, to obtain a systolic lower bound for the total volume corresponding to the partition

$$n = 1 + 1 + \cdots + 1 + (n-b),$$

where the summand 1 occurs b times. \square

Our goal is to describe a sufficient condition for applying Gromov’s theorem, so as to obtain such a lower bound in the case when the fiber class in M vanishes. From now on we assume that M is orientable, has dimension n , and $b_1(M) = 2$. Let $\{\alpha, \beta\} \subset H^1(M)$ be a basis of $H^1(M)$. Let F_M be a typical fiber of the Abel–Jacobi map. It is easy to see that $[F_M]$ is Poincaré dual to the cup product $\alpha \cup \beta$. Thus, if $\alpha \cup \beta \neq 0$ then $\text{cat}_{\text{sys}} M \geq 3$ by Proposition 7.2. If $\alpha \cup \beta = 0$ then the Massey product $\langle \alpha, \alpha, \beta \rangle$ is defined and has zero indeterminacy.

7.3. Theorem. *If $\langle \alpha, \alpha, \beta \rangle \neq 0$ then $\text{cat}_{\text{sys}} M \geq 3$.*

Note that also $\text{cat}_{\text{LS}} M \geq 3$ if $\langle \alpha, \alpha, \beta \rangle \neq 0$, since $\langle \alpha, \alpha, \beta \rangle$ has category weight 2 [Ru99].

To prove the theorem, we reformulate it in the dual homology language.

7.4. Definition. Let $F = F_M \subset M$ be an oriented typical fiber. Assume $[F] = 0 \in H_{n-2}(M)$. Choose an $(n-2)$ -chain X with $\partial X = F$. Consider another regular fiber $F' \subset M$. The oriented intersection $X \cap F'$ defines a class

$$\ell_M(F_M, F_M) \in H_{n-3}(M),$$

which will be referred to as the *self-linking class* of a typical fiber of \mathcal{A}_M .

The following lemma asserts, in particular, that the self-linking class is well-defined, at least up to sign.

7.5. Lemma. *The class $\ell_M(F_M, F_M)$ is dual, up to sign, to the cohomology class $\langle \alpha, \alpha, \beta \rangle \cup \beta \in H^3(M)$.*

Proof. The classes α, β are Poincaré dual to hypersurfaces $A, B \subset M$ obtained as the inverse images under \mathcal{A}_M of a pair $\{u, v\}$ of loops defining a generating set for \mathbb{T}^2 . Clearly, the intersection $A \cap B \subset M$ is a typical fiber

$$F_M = A \cap B$$

of the Abel-Jacobi map (namely, inverse image of the point $u \cap v \in \mathbb{T}^2$). Then another regular fiber F' can be represented as $A' \cap B'$ where, say, the set A' is the inverse image of a loop u' “parallel” to u . Then $A' \cap X$ is a cycle, since $\partial(A' \cap X) = A' \cap A \cap B = \emptyset$. Moreover, it is easy to see that the homology class $[A' \cap X]$ is dual to the Massey product $\langle \alpha, \alpha, \beta \rangle$. (We take a representative a of α such that $a \cup a = 0$.) Now, since $F' = A' \cap B'$, we conclude that $[F' \cap X]$ is dual, up to sign, to $\langle \alpha, \alpha, \beta \rangle \cup \beta$. \square

7.6. Remark. In the case of 3-manifolds with Betti number 2, the non-vanishing of the self-linking number is equivalent to the non-vanishing of C. Lescop’s generalization λ of the Casson-Walker invariant, cf. [Le96]. See T. Cochran and J. Masters [CM05] for generalizations.

Now Theorem 7.3 will follow from Theorem 7.7 below.

7.7. Theorem. *If the self-linking class in $H_{n-3}(M)$ is non-trivial, then $\text{cat}_{\text{sys}}(M) \geq 3$.*

The theorem is immediate from the proposition below. If the fiber class in M of the Abel-Jacobi map vanishes, one can define the self-linking of a typical fiber, and proceed as follows.

7.8. Proposition. *The non-vanishing of the self-linking of a typical fiber $\mathcal{A}_M^{-1}(p)$ of $\mathcal{A}_M : M \rightarrow \mathbb{T}^2$ is a sufficient condition for the non-vanishing of the fiber class $[\overline{F}_M]$ in the maximal free abelian cover \overline{M} of M .*

Proof. The argument is modeled on the one found in [KL05] in the case of 3-manifolds, and due to A. Marin (see also [Ka07, p. 165-166]).

Consider the pullback diagram

$$\begin{array}{ccc} \overline{M} & \xrightarrow{\overline{\mathcal{A}}_M} & \mathbb{R}^2 \\ p \downarrow & & \downarrow \\ M & \xrightarrow{\mathcal{A}_M} & \mathbb{T}^2 \end{array}$$

where \mathcal{A}_M is the Abel–Jacobi map and the right-hand map is the universal cover of the torus. Take $x, y \in \mathbb{R}^2$. Let $\overline{F}_x = \overline{\mathcal{A}}_M^{-1}(x)$ and $\overline{F}_y = \overline{\mathcal{A}}_M^{-1}(y)$ be lifts of the corresponding fibers $F_x, F_y \subset M$. Choose a properly imbedded ray $r_y \subset \mathbb{R}^2$ joining the point $y \in \mathbb{R}^2$ to infinity while avoiding x (as well as its \mathbb{Z}^2 -translates), and consider the complete hypersurface

$$S = \overline{\mathcal{A}}^{-1}(r_y) \subset \overline{M}$$

with $\partial S = \overline{F}_y$. We have $S \cap g.\overline{F}_x = \emptyset$ for all $g \in G$, where $G = \mathbb{Z}^2$ denotes the group of deck transformations of the covering $p : \overline{M} \rightarrow M$.

We will prove the contrapositive. Namely, the vanishing of the class of the lift of the fiber implies the vanishing of the self-linking class. If the surface \overline{F}_x is zero-homologous in \overline{M} , we can choose a compact hypersurface $\Sigma \subset \overline{M}$ with

$$\partial \Sigma = \overline{F}_x.$$

The linking $\ell_M(F_x, F_y)$ in M can therefore be computed as the oriented intersection

$$\begin{aligned} \ell_M(F_x, F_y) &= p(\Sigma) \cap F_y \\ &= \sum_{g \in G} g.\Sigma \cap \overline{F}_y \\ (7.1) \quad &= \sum_{g \in G} \partial(g.\Sigma \cap S) \\ &\sim 0, \end{aligned}$$

where all sums and intersections are oriented, and the intersection surface $\Sigma \cap S$ is compact by construction. \square

To summarize, if the lift of a typical fiber to the maximal free abelian cover of M^n with $b_1(M) = 2$ defines a nonzero class, then one obtains the lower bound $\text{cat}_{\text{sys}}(M) \geq 3$, due to the existence of a suitable systolic inequality corresponding to the partition

$$n = 1 + 1 + (n - 2)$$

as in (6.1), by applying Gromov's Theorem 7.1.

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